Modular quantum dilogarithm, hyperbolic beta integrals and integrable systems

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In 1994 Faddeev introduced a **modular quantum dilogarithm** using the simplest $SL(2,\mathbb{Z})$ group transformation

$$u \to \frac{u}{\tau}, \qquad \tau \to -\frac{1}{\tau},$$

applied to the q-Pochhammer symbol

$$(e^{2\pi i u}; q)_\infty := \prod_{j=0}^{\infty}(1 - e^{2\pi i u} q^j), \qquad q := e^{2\pi i \tau}.$$

In the theory of special functions = **the hyperbolic gamma function**. Its generalization based on arbitrary $SL(2,\mathbb{Z})$-transformation: Dimofte, 2015.

The aim: to present the evaluation formula for a general univariate hyperbolic beta integral built with the help of this function and indicate possible applications of the derived identity.

Joint with G. A. Sarkissian, arXiv:1910.11747 [hep-th], Proc. Steklov Inst. of Math., v. 309 (2020)

Classical automorphic objects.

$(z;q)_\infty := \prod_{j=0}^\infty (1-zq^j), \quad |q|<1.$

The Dedekind η-function

$$\eta(\tau) := e^{\frac{\pi i\tau}{12}}(e^{2\pi i\tau};e^{2\pi i\tau})_\infty$$

and Jacobi θ_1-function

$$\theta_1(u|\tau) = -\theta_{11}(u) := -\sum_{\ell\in\mathbb{Z}+1/2} e^{\pi i\tau\ell^2}e^{2\pi i\ell(u+1/2)}$$

$$= iq^{1/12}e^{-\pi iu}\,\eta(\tau)\,\theta(e^{2\pi iu};q),$$

$$\theta(z;q) := (z;q)_\infty(qz^{-1};q)_\infty.$$

Trigonometric q-gamma function

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \underset{q\to 1^-}{\to} \Gamma(x).$$

The relation

$$\Gamma_q(x)\Gamma_q(1-x) \propto \frac{1}{\theta_1(x|\tau)}$$

is an analog of $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$.

Let $\quad M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.$

Modular transformation laws for $c > 0$

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d)\sqrt{-i(c\tau + d)}\,\eta(\tau),$$

$$\theta_1\left(\frac{u}{c\tau + d}\middle|\frac{a\tau + b}{c\tau + d}\right) = \frac{1}{i}\varepsilon(a, b, c, d)^3\sqrt{\frac{c\tau + d}{i}}\, e^{\frac{\pi i c u^2}{c\tau + d}}\,\theta_1(u|\tau),$$

where the character (a 24-th root of unity)

$$\varepsilon = \varepsilon(a, b, c, d) := \begin{cases} \left(\frac{d}{c}\right) e^{\frac{\pi i(1-c)}{4}} e^{\frac{\pi i}{12}[bd(1-c^2)+c(a+d)]} & \text{for odd}\, c, \\ \left(\frac{c}{d}\right) e^{\frac{\pi i d}{4}} e^{\frac{\pi i}{12}[ac(1-d^2)+d(b-c)]} & \text{for odd}\, d. \end{cases}$$

Here $\left(\frac{d}{c}\right)$ is the Legendre-Jacobi symbol. For odd $c > 0$

$$\left(\frac{d}{c}\right) = (-1)^{g_c(d)}, \quad g_c(d) = \sum_{\ell=1}^{(c-1)/2}\left[\frac{2d\ell}{c}\right],$$

$[x]$ = integer part of $x \in \mathbb{R}$; $\left(\frac{d}{c}\right) = 1$ if $d = x^2 \mod c$, c prime.

Equivalently,

$$\varepsilon(a,b,c,d) = e^{-\pi i S(d,c)} e^{\frac{\pi i}{12c}(a+d)},$$

where $S(d,c)$ is the Dedekind sum

$$S(d,c) = \sum_{j=1}^{c-1} \frac{j}{c} \left(\frac{dj}{c} - \left[\frac{dj}{c} \right] - \frac{1}{2} \right).$$

The triple product Jacobi identity $\Rightarrow$

$$\frac{\theta(e^{-\frac{2\pi iu}{c\tau+d}}; e^{2\pi i\frac{a\tau+b}{c\tau+d}})}{\theta(e^{2\pi iu}; e^{2\pi i\tau})} = \frac{(e^{2\pi i\frac{a\tau+b+u}{c\tau+d}}; e^{2\pi i\frac{a\tau+b}{c\tau+d}})_\infty}{(e^{2\pi iu}; e^{2\pi i\tau})_\infty} \frac{(e^{-\frac{2\pi iu}{c\tau+d}}; e^{2\pi i\frac{a\tau+b}{c\tau+d}})_\infty}{(e^{2\pi i(\tau-u)}; e^{2\pi i\tau})_\infty}$$

$$= i\varepsilon^2 \, e^{\frac{\pi i}{6}(\tau - \frac{a\tau+b}{c\tau+d})} \, e^{-\pi iu(1+\frac{1}{c\tau+d})} \, e^{\frac{\pi icu^2}{c\tau+d}}. \qquad (*)$$

Again, an analog of $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$ **!**

The trick. Let us parametrize

$$\tau := \frac{\omega_1 - d\omega_2}{c\omega_2}, \quad \tilde{\tau} := \frac{a\tau + b}{c\tau + d} = \frac{a\omega_1 - \omega_2}{c\omega_1}, \quad \tilde{q} = e^{2\pi i\tilde{\tau}}.$$

Then $c\tau + d = \omega_1/\omega_2$. For $\mu \in \mathbb{C},\ m \in \mathbb{Z}_c = \{0, 1, \ldots, c-1\}$,

$$\gamma_M(\mu, m) = \gamma_M(\mu, m; \omega_1, \omega_2) := \frac{(\tilde{q}e^{2\pi i\frac{\mu + am\omega_1}{c\omega_1}}; \tilde{q})_\infty}{(e^{2\pi i\frac{\mu + m\omega_2}{c\omega_2}}; q)_\infty}.$$

For $a = d = 0$, $c = -b = 1 \Rightarrow \tilde{\tau} = -1/\tau$. Faddeev (1994): modular quantum dilogarithm (hyperbolic gamma function)

$$\gamma(\mu; \omega_1, \omega_2) := \frac{(\tilde{q}e^{2\pi i\frac{\mu}{\omega_1}}; \tilde{q})_\infty}{(e^{2\pi i\frac{\mu}{\omega_2}}; q)_\infty} = \exp\left[-\int_{\mathbb{R}+i0} \frac{e^{\mu x} dx/x}{(e^{\omega_1 x} - 1)(e^{\omega_2 x} - 1)}\right].$$

Well defined for $\omega_1/\omega_2 > 0$, i.e. $|q| = 1$.

General case: $\tilde{\tau} = \frac{a\tau + b}{c\tau + d}$ Dimofte, 2015

Good motivation: compute partition function of the Chern-Simons theory on the $3d$ squashed lens space $L(c, a)_\tau$:

$$\kappa^2|z_1|^2 + \kappa^{-2}|z_2|^2 = 1, \quad (z_1, z_2) \cong (e^{\frac{2\pi i}{c}} z_1, e^{\frac{2\pi i a}{c}} z_2), \quad \kappa = \kappa(\tau).$$

Define the normalized modular dilogarithm (S. & S., 2019)

$$\Gamma_M(\mu, m) := Z(m) e^{-\frac{\pi i}{2c} B_{2,2}(\mu;\omega_1,\omega_2)} \gamma_M(\mu, m),$$

$$Z(m) = \frac{e^{\frac{\pi i}{4}(\frac{a+d+3}{3c}-1)}}{\varepsilon(a,b,c,d)} e^{\pi i \frac{(1+b)c+a}{2c} m(m+d+1)},$$

$$B_{2,2}(\mu;\omega_1,\omega_2) = \frac{1}{\omega_1\omega_2}\left((\mu - \frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12}\right),$$

the multiple Bernoulli polynomials of the 2nd order.

Then modular transformation law (*) can be written in the form

$$\Gamma_M(\omega_1+\omega_2-\mu, -d-1-m)\Gamma_M(\mu, m) = 1$$

= reflection equation for the “rarefied” hyperbolic gamma function.

Some properties of this generalized gamma function, complimenting the theory of Jacobi forms.

Denote $2\pi i\omega_1/\omega_2 = -\delta$ and take $\delta \to 0^+$.
Then $q \to \epsilon$, $\epsilon = e^{-2\pi i d/c}$, $\epsilon^c = 1$ and $\tilde{q} \to 0 \Rightarrow$

$$\log \gamma_M(\mu, m) = \sum_{n=1}^{\infty} \Big(\frac{e^{2\pi i n \frac{\mu + m\omega_2}{c\omega_2}}}{n(1-q^n)} - \frac{\tilde{q}^n e^{2\pi i n \frac{\mu + am\omega_1}{c\omega_1}}}{n(1-\tilde{q}^n)} \Big)$$

$$\underset{\delta\to 0}{\propto} \frac{1}{c\delta} \mathrm{Li}_2(e^{2\pi i \frac{\mu}{\omega_2}}), \quad \mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad \text{the dilogarithm function.}$$

Residue relation and difference equations:

$$\lim_{\mu\to 0} \mu\, \Gamma_M(\mu, 0) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\, c,$$

$$\frac{\Gamma_M(\mu + \omega_1, m - d)}{\Gamma_M(\mu, m)} = (-1)^m e^{\pi i \frac{(b+1)(d+1)}{2}} 2 \sin \frac{\pi(\mu + m\omega_2)}{c\omega_2},$$

where $e^{\pi i (b+1)(d+1)} = 1$, and

$$\frac{\Gamma_M(\mu + \omega_2, m - 1)}{\Gamma_M(\mu, m)} = (-1)^{(b+1)m} e^{\pi i \frac{(b+1)(d+1)}{2}} 2 \sin \frac{\pi(\mu + am\omega_1)}{c\omega_1}.$$

Main result (S. & S., 2019). Theorem. Let $\mathrm{Re}(\omega_{1,2})>0$, for $j=1,\ldots,6$ take $a_j\in\mathbb{C}$, $\mathrm{Re}(a_j)>0$, $n_j\in\mathbb{Z}+\nu$, $\nu=0,\frac{1}{2}$ with

$$\sum_{j=1}^{6} a_j=\omega_1+\omega_2, \qquad \sum_{j=1}^{6} n_j=-d-1.$$

Then $\Rightarrow$ the most general known hyperbolic beta integral

$$\sum_{m\in\mathbb{Z}_c+\nu}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{6}\Gamma_M(a_j\pm\mu,n_j\pm m)}{\Gamma_M(\pm 2\mu,\pm 2m)}\frac{d\mu}{2ic\sqrt{\omega_1\omega_2}}$$
$$=\prod_{1\le\ell<j\le 6}\Gamma_M(a_\ell+a_j,n_\ell+n_j).$$

Compact notation

$$\Gamma_M(g\pm\mu,n\pm m):=\Gamma(g+\mu,n+m)\Gamma(g-\mu,n-m).$$

Equivalently, for $N:=\sum_{1\le\ell<j\le 6}n_\ell n_j$:

$$\sum_{m\in\mathbb{Z}_c+\nu}\int_{-i\infty}^{i\infty}e^{\frac{2\pi i a}{c}(m^2-\nu^2)}e^{-\frac{2\pi i}{c\omega_1\omega_2}\mu^2}\frac{\prod_{j=1}^{6}\gamma_M(a_j\pm\mu,n_j\pm m)}{\gamma_M(\pm 2\mu,\pm 2m)}\frac{d\mu}{2ic\sqrt{\omega_1\omega_2}}$$
$$=e^{-2\pi i\frac{(1+b)c+a}{c}\left(N+\nu^2\right)}e^{\pi i\left(\frac{1}{c\omega_1\omega_2}(\frac{7}{12}(\omega_1+\omega_2)^2-\sum_{j=1}^{6}a_j^2)-\frac{5}{4}(1-\frac{2}{3c})+5S(d,c)\right)}$$
$$\times\prod_{1\le\ell<j\le 6}\gamma_M(a_\ell+a_j,n_\ell+n_j).$$

An analog of Euler's beta integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Applications:

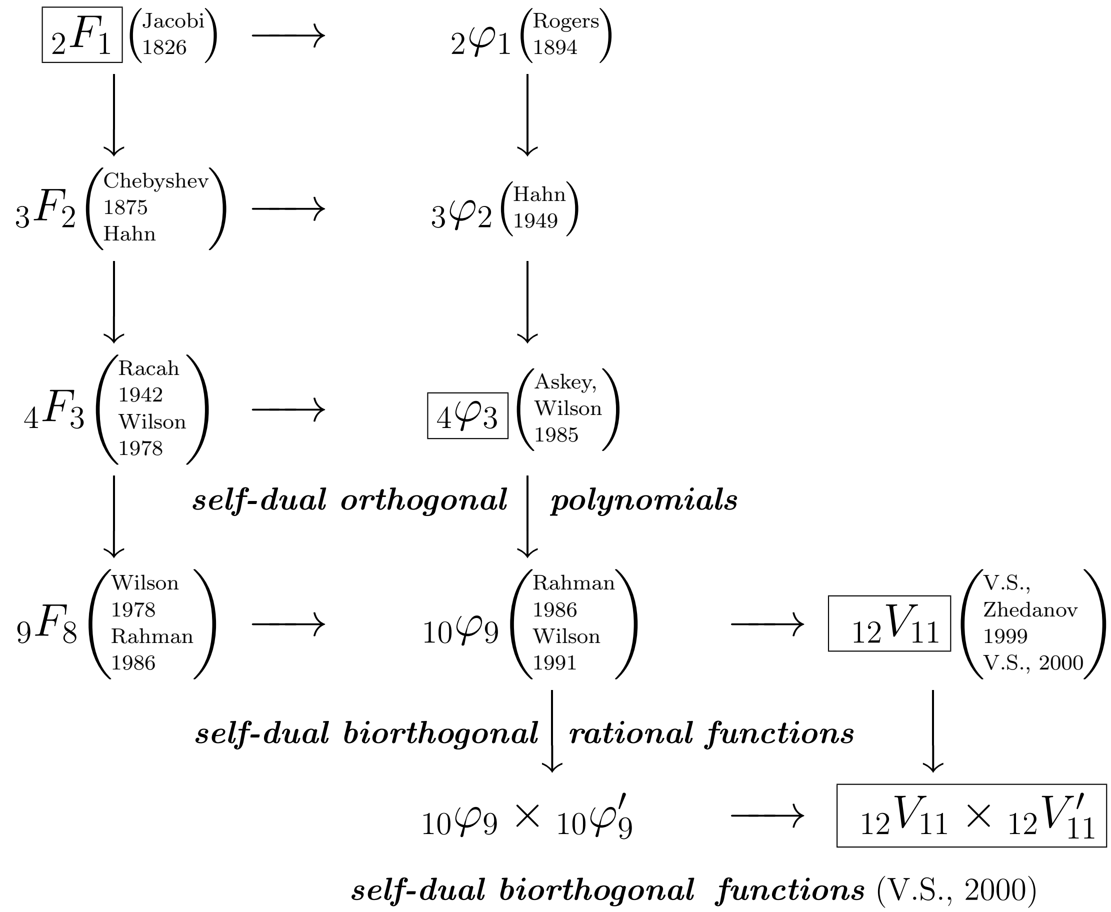
- **To do:** solution of the star-triangle relation $\Rightarrow$ solution of YBE. In general the Boltzmann weights are not positive.

- **To do:** identify via localization l.h.s. with the PF (partition function) of a $3d$ supersymmetric field theory having $G = SU(2)$, $F = SU(6)$ and chiral superfield in the fundamental representation of F living in general squashed lens space $L(c, a)_\tau$ and r.h.s. with the PF of dual confined theory.

- **To do:** the formula should be related to fusion matrices in generalized parafermionic $2d$ CFT (generalized para-Liouville theory) of Bonelli, Maruyoshi, Tanzini, Yagi, 2013

- **To do:** l.h.s. should define the biorthogonality measure for univariate special functions generalizing ${}_{10}\varphi_9 \times {}_{10}\varphi_9'$ functions and related q-analog of the ${}_2F_1$ Euler-Gauss hypergeometric function. The latter should define special eigenfunctions of $N = 1$ Hamiltonian of some N-body Ruijsenaars type integrable system.

CLASSICAL ORTHOGONAL POLYNOMIALS AND THEIR GENERALIZATIONS



Open problem:

an extension of the Faddeev modular double.

Faddeev, 2000: $U_q(sl_2) \to U_q(sl_2) \times U_{\tilde{q}}(sl_2),\ q = e^{2\pi i\tau}, \tilde{q} = e^{-2\pi i\frac{1}{\tau}}$.

Now: an "$SL(2,\mathbb{Z})$" extension, $\tilde{q} = e^{2\pi i\frac{a\tau+b}{c\tau+d}}$.

One should get a novel ("arithmetic") quantum algebra

$$U_{\tau,a,c}(sl_2), \quad q \to \tau, \quad c \in \mathbb{Z}_{>0}, \quad a \in \mathbb{Z}_c.$$

Defining relations ? In progress, ...