BETWEEN THE RUDIN–KEISLER AND COMFORT PREORDERS

TOPICS: SET THEORY, GENERAL TOPOLOGY

NIKOLAI L. POLIAKOV, DENIS I. SAVELIEV

We consider ultrafilters over ω (although most of our results remain true for ultrafilters over any infinite set); as usual, $\beta\omega$ denotes the set of them. For $\mathfrak{u}, \mathfrak{v} \in \beta\omega$ and any ordinal α, define: $\mathfrak{u}\, R_0\, \mathfrak{v}$ iff $\mathfrak{u}$ is principal, $R_{<\alpha} = \bigcup_{\beta<\alpha} R_\beta$, and $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous map $f : \beta\omega \to \beta\omega$ such that $f(\mathfrak{v}) = \mathfrak{u}$ and $f(n)\, R_{<\alpha}\, \mathfrak{v}$ for all $n < \omega$. The hierarchy is non-degenerate and lies between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$, the Rudin–Keisler and Comfort preorders.

Theorem 1. *$R_1 = \leq_{\mathrm{RK}}$; $R_{<\alpha} \subset R_\alpha$ for all $\alpha < \omega_1$; $R_{<\omega_1} = R_{\omega_1} = \leq_{\mathrm{C}}$.*

If $n < \omega$, the relations R_n can be redefined in terms of right-continuous ultrafilter extensions of n-ary operations on ω as follows: $\mathfrak{u}\, R_n\, \mathfrak{v}$ iff there exists $f : \omega^n \to \omega$ such that $\widetilde{f}(\mathfrak{v}, \ldots, \mathfrak{v}) = \mathfrak{u}$. Moreover, $R_m \circ R_n = R_{nm}$ (so R_n are not preorders for $2 \leq n < \omega$). These observations can be expanded to all R_α by using ω-ary operations. Such an operation is identified with a continuous map of the Baire space ω^ω into the discrete space ω; these maps admit a natural hierarchy ranked by countable ordinals. Any continuous $f : \omega^\omega \to \omega$ uniquely extends to a right-continuous $\widetilde{f} : (\beta\omega)^\omega \to \beta\omega$, i.e., an ω-ary operation on $\beta\omega$.

Proposition 1. *Let $\alpha < \omega_1$ and $\mathfrak{u}, \mathfrak{v} \in \beta\omega$. Then $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous $f : \omega^\omega \to \omega$ of rank α such that $\widetilde{f}(\mathfrak{v}, \mathfrak{v}, \ldots) = \mathfrak{u}$.*

The composition of arbitrary $R_{<\alpha}$ is expressed via a multiplication-like operation on ordinals. To simplify notation, denote $\sup_{\gamma<\alpha}(\gamma \cdot \beta)$ by $({<}\alpha) \cdot \beta$; the explicit calculation of these ordinals, used in getting the following result, is rather cumbersome.

Theorem 2. *Let $\alpha, \beta < \omega_1$.*

- (i) *$R_\alpha \circ R_\beta = R_\gamma$ where $\gamma = \beta \cdot \alpha$ if $\beta = 0$ or $\alpha < \omega$, $\gamma = \beta \cdot (\alpha + 1) - 1$ if $0 < \beta < \omega$ and $\alpha \geq \omega$, and $\gamma = \beta \cdot (\alpha + 1)$ if $\alpha, \beta \geq \omega$;*
- (ii) *If $\alpha > 0$ is limit, then $R_{<\alpha} \circ R_\beta = R_{<\gamma}$ where $\gamma = \beta \cdot \alpha$;*
- (iii) *If $\beta > 0$ is limit, then $R_\alpha \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$ if $\alpha < \omega$, and $\gamma = ({<}\beta) \cdot (\alpha + 1)$ otherwise;*
- (iv) *If $\alpha, \beta > 0$ are limit, then $R_{<\alpha} \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$.*

Corollary 1. *Let $2 \leq \alpha \leq \omega_1$. Then $R_{<\alpha}$ is a preorder iff α is multiplicatively indecomposable.*

Define preorders between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ by letting $\leq_0 = \leq_{\mathrm{RK}}$ and $\leq_{1+\alpha} = R_{<\omega^{\omega^\alpha}}$ for all $\alpha \leq \omega_1$. So, if α is infinite, $R_{<\alpha} = \leq_\alpha$ iff α is an epsilon number. Also $\leq_\alpha \circ \leq_\beta = \leq_\gamma$ where $\gamma = \max(\alpha, \beta)$.

As was known, for any ultrafilter $\mathfrak{v}$ and semigroup S, the set $\{\mathfrak{u} : \mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v}\}$ forms a subsemigroup of βS. This can be expanded to arbitrary first-order models and relations $R_{<\alpha}$ as follows.

Proposition 2. *For every $\alpha > 1$, ultrafilter $\mathfrak{v}$, and model $\mathfrak{A}$ of any signature, $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ forms a submodel of the model $\beta\mathfrak{A}$ iff α is additively indecomposable. Consequently, for all α, $\mathfrak{v}$, and $\mathfrak{A}$, $\{\mathfrak{u} : \mathfrak{u} \leq_\alpha \mathfrak{v}\}$ forms a submodel of $\beta\mathfrak{A}$.*

Ultrafilter extensions of ω-ary operations can be used to state Ramsey-type results. Let $f[X]$ be the image of X under f, and let $I = \{x \in \omega^\omega : x \text{ is increasing}\}$. If $X \subseteq \omega$ and $f : \omega^\omega \to Y$, we say that f is *constant upward on* X iff $|f[X \cap I]| = 1$, and *quasi-invertible upward on* X iff there exists $g : Y \to \omega$ such that for any infinite $A \subseteq X$ we have $g[f[A^\omega \cap I]] \subseteq A$ and $|A \setminus g[f[A^\omega \cap I]]| < \omega$. The following refines the characterization of Ramsey ultrafilters as selective ones.

Theorem 3. *Let $\mathfrak{u} \in \beta\omega$. Then $\mathfrak{u}$ is $\leq_{\mathrm{RK}}$-minimal iff any continuous $f : \omega^\omega \to \omega$ is either constant upward or quasi-invertible upward on some $X \in \mathfrak{u}$.*

Nikolai L. Poliakov, HSE University, niknikols0@gmail.com; Denis I. Saveliev (corresponding author): Institute for Information Transmission Problems RAS, d.i.saveliev@gmail.com.