НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Aggregation of preferences: impossibilities and possibilities

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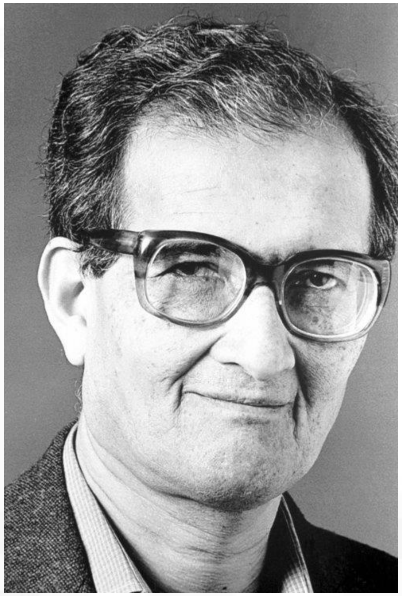
Motivation

Choices of a rational individual are guided, explained and predicted by his/her *preferences*.

Since collective choices are ubiquitous, how to define preferences of a rational **collective** actor?

Dedicated to the Founding Fathers of Social Choice

Kenneth Arrow
1921 - 2017
Nobel Memorial Prize in Economics 1972



Amartya Sen
b. 1933
Nobel Memorial Prize in Economics 1998

Individual choice

A – the *general set* of all possible options (alternatives).

Supposition: A is finite.

X – the *feasible set* of alternatives: $X \subseteq A \wedge X \neq \varnothing$.The feasible set is a variable.

A choice is a subset C of X: $C \subseteq X$.

Supposition: an actor always chooses the same subset C from X.

Consequently, choices are representable by a *choice function* $C(X)$.

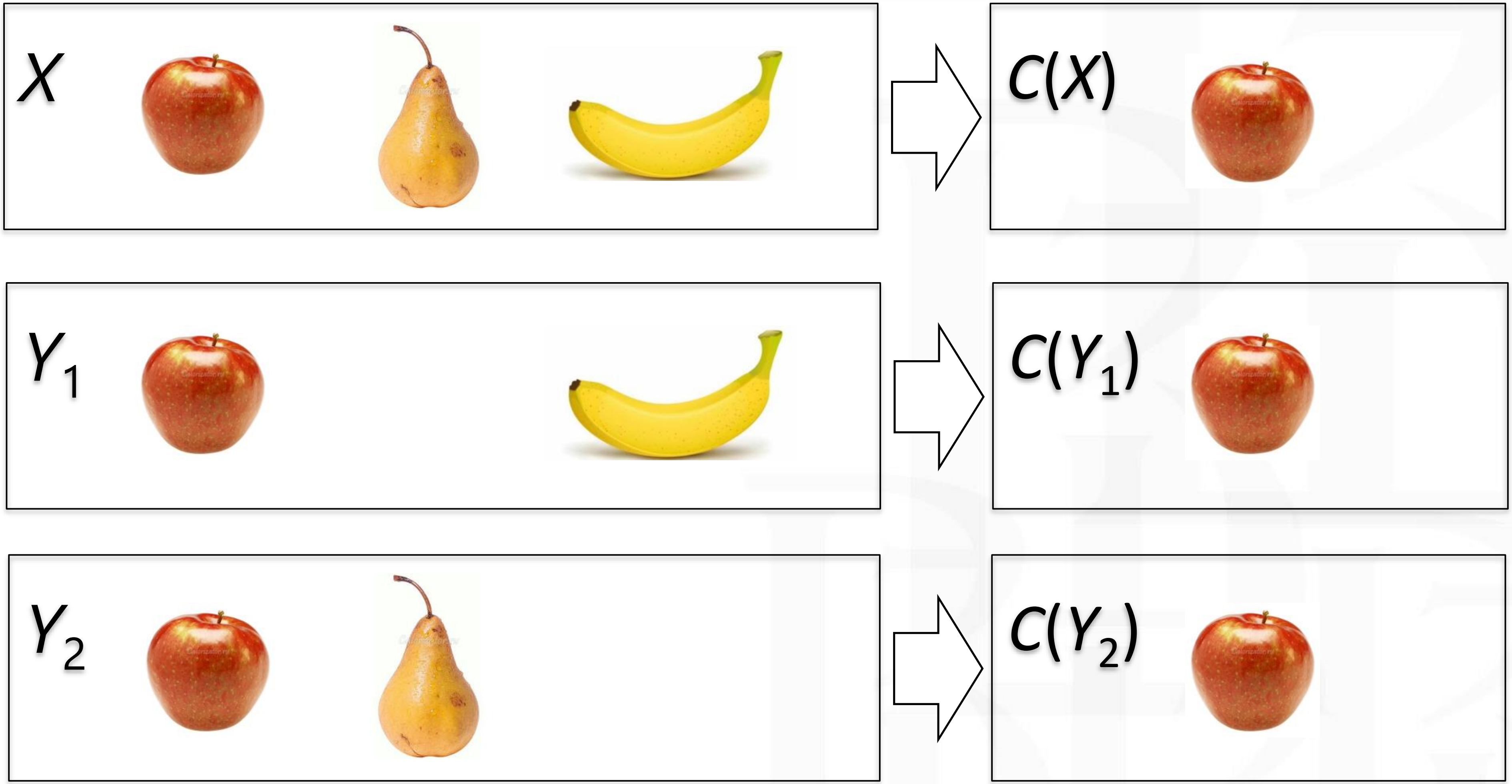
Supposition: $C(X)$ of a *rational* actor satisfy the following axioms.

- ***Nonemptiness***: $\forall\ X \subseteq A,\ C(X) \neq \varnothing$.
- ***Nash Independence of irrelevant alternatives*** (Nash 1950):

$$\forall\ X \subseteq A,\ \forall\ Y \subseteq X,\ C(X) \cap Y \neq \varnothing \Rightarrow C(Y) = C(X) \cap Y$$

Nash independence of irrelevant alternatives (NIIA)

$$\forall\, X \subseteq A,\ \forall\, Y \subseteq X,\ C(X) \cap Y \neq \varnothing \Rightarrow C(Y) = C(X) \cap Y$$



Preferences

A *ranking R* of alternatives from *A* is *a weak ordering* of *A*,

that is, a *binary relation* $R \subseteq A\times A$ satisfying two axioms:

- ***Completeness***: all alternatives are comparable, $\forall\ x, y \in A,\ xRy \vee yRx$;
- ***Transitivity***: $\forall\ x, y, z \in A,\ (xRy \wedge yRz) \Rightarrow xRz$.

Theorem: $C(X)$ satisfies nonemptiness and Nash IIA if and only if there is a (unique) ranking *R* of alternatives from *A* such that $C(X)=MAX(R|_X)$ for any *X*.

$R|_X = R\cap X\times X$ – the *restriction* of a relation *R* ($R\subseteq A\times A$) onto a subset *X* ($X\subseteq A$).

$MAX(R|_X)=\{a \in X \mid \forall\ b \in X,\ bRa \Rightarrow aRb\}$ - *the set of maximal elements* of $R|_X$

That is, a rational agent is a maximizing agent, and *R* represents his/her *preferences* that rationalize (i.e. make understandable) his/her choices $C(X)$.

Utility

Since A is finite, any ranking R (and only a ranking) of A can be represented by a real-valued function $u(x)$: $A \to \mathbb{R}$, such that $u(y) \geq u(x) \Leftrightarrow yRx$.

$u_0=u(x)$ is called the *utility* that a rational agent with preferences R derives from alternative x. Correspondingly, $u(x)$ is called his/her *utility function*.

What is the meaning of $u(x)$?

- If the utility is either unobservable, or unmeasurable, or lacking proper definition, or simply fictitious, then $u(x)$ is no more than a convenient mathematical representation of a ranking.
- If the utility is observable and measurable on some ordinal or cardinal scale, then $u(x)$ is the result its measurement or estimation. When the ranking is based on such an evaluation, it is called ***rating***.

Rankings and ratings (examples)

Ranking

x (presidential candidates)	*u*(*x*)
a	5
b c d	4
f g	3

No scale

Ordinal rating

x (students)	*u*(*x*)
a	excellent
b c d	good
f g	satisfactory

Ordinal scale

Cardinal rating

x (investment projects)	*u*(*x*)
a	5100$
b c d	4300$
f g	3700$

Cardinal scale

Interpersonal comparisons of utilities

If the utility of an agent is unmeasurable or ordinally measurable then it is possible to replace $u(x)$ with $u'(x)=\varphi(u(x))$, where transformation φ is an arbitrary monotonically and strictly increasing function $\varphi: \mathbb{R} \to \mathbb{R}$.

Similarly, if the utility of an agent is cardinally measurable, and if neither the unit nor the origin of the scale are fixed, then it is possible to replace $u(x)$ with its *affine* transform $u'(x)=a*u(x)+b$, where a and b are arbitrary real numbers ($a>0$).

If there exist no common scale of measurement of utility, either cardinal or ordinal, utility functions may be transformed independently, which means the utilities that agents derive from any given state are **incomparable**.

As a result, all claims like "Ann will benefit from alternative x (i.e. from the state of nature when x is chosen) more than Bob" will be **unverifiable**.

Axioms of ordinal comparability and noncomparability

- **Ordinal noncomparability (ONC)**

Any set of utility functions $\{u_k(x)\}$, $k=1\div n$, in all computations can be replaced by a set $\{u'_k(x)\}$, $k=1\div n$, $u'_k(x)=\varphi_k(u_k(x))$, where transformations φ_k are arbitrary monotonically and strictly increasing functions φ_k: $\mathbb{R}\to\mathbb{R}$.

- **Ordinal comparability (OC)**

Any set of utility functions $\{u_k(x)\}$, $k=1\div n$, in all computations can be replaced by a set $\{u'_k(x)\}$, $k=1\div n$, $u'_k(x)=\varphi(u_k(x))$, where transformation φ is an arbitrary monotonically and strictly increasing functions φ: $\mathbb{R}\to\mathbb{R}$

(OC) implies the existence of a common **ordinal** scale of measurement. This scale allows one to make **ordinal** interpersonal comparisons of utilities. In such a setting the claim "Ann is getting more than Bob" is verifiable.

Axioms of cardinal comparability and noncomparability

- **Cardinal noncomparability (ONC)**

Any set of utility functions $\{u_k(x)\}$, $k=1\div n$, in all computations can be replaced by a set $\{u'_k(x)\}$, $k=1\div n$, $u'_k(x)=a_k*u(x)+b_k$, where a_k and b_k are $2*n$ arbitrary real numbers ($a_k>0$ for all k).

- **Cardinal comparability (OC)**

Any set of utility functions $\{u_k(x)\}$, $k=1\div n$, in all computations can be replaced by a set $\{u'_k(x)\}$, $k=1\div n$, $u'_k(x)=\varphi(u_k(x))$, $a*u(x)+b$, where a and b are two arbitrary real numbers ($a>0$).

(CC) implies the existence of a common **cardinal** scale of measurement. This scale allows one to make **both** ordinal and cardinal interpersonal comparisons of utilities.

Social choice

A – the (finite) general set of *social* alternatives (possible states of the world)

X – the feasible set: $X \subseteq A \wedge X \neq \varnothing$.

N – the *society* (e.g. a board of directors, a constituency of voters, a panel of experts)

$u_k(x)$ – the utility of social alternative $x \in A$ for voter $k \in N$

$U = \{ u_k(x) \mid k \in N \}$ – a *profile* of utility functions

Problem: Given U define either $R=R(U)$ or $P=P(U)$.

R – (*weak*) *social preferences*, $R \subseteq A \times A$

P – *strict social preferences*, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$

Aggregation rule $R=R(U)$ is called *social welfare functional* (Sen 1970)

Multi-criteria choice

A – the (finite) general set of alternatives (e.g. journals, countries)

N – the set of *criteria* (various indicators)

$u_k(x)$ – the value of criterion $k \in N$ for alternative $x \in A$

$U = \{ u_k(x) \mid k \in N \}$ – a *profile* of criterial evaluations

Problem: Given U define a ranking $R=R(U)$.

Journal	IF	5-IF	Immediacy index	Article influence	Hirsch	SNIP	SJR
Explorations in Economic History	0.935	0.898	0.541	0.772	7	1.768	0.036
Review of Income and Wealth	0.805	1.103	0.205	0.850	9	1.712	0.034
Scandinavian Journal of Economics	0.514	1.070	0.150	1.310	8	1.426	0.043

Axioms of aggregation. Arrow's impossibility theorem

- ***Full domain***: the rule can be applied in all cases, i.e. to any utility profile U.
- ***Completeness***: all alternatives are comparable, $\forall\ x, y \in A,\ xRy \vee yRx$.
- ***Transitivity***: $\forall\ x, y, z \in A,\ (xRy \wedge yRz) \Rightarrow xRz$.
- ***Weak Pareto principle***: if $\forall k \in N,\ u_k(x) \geq u_k(y)$, then xRy and

  if $\forall k \in N,\ u_k(x) > u_k(y)$, then xPy.
- ***Independence of irrelevant utilities***: $\forall X \subseteq A,\ R(U)|_X = R(U|_X)$

$$(\text{FD}) \wedge (\text{C}) \wedge (\text{T}) \wedge (\text{WP}) \wedge (\text{IIU}) \Rightarrow \text{Neutrality}$$

- ***Neutrality***: the rule treats all alternatives equally.
- ***Ordinal Noncomparability***

Arrow's theorem (1950): If $|A|>2$, then the only such rule is a **dictatorship**.

$$\exists d \in N : R = R_d$$

Impossibility 2

- ***Full domain***
- ***Completeness***
- ***Transitivity***
- ~~***Weak Pareto principle***~~
- ***Independence of irrelevant utilities***
- ***Neutrality***
- ***Ordinal Noncomparability***

Theorem (Wilson, 1972): If $|A|>2$, then there must be

either a dictator, $\exists d \in N$: $R=R_d$

or an antidictator, $\exists d \in N$: $\forall\ x, y \in A$, $xRy \Leftrightarrow yR_dx$.

Impossibility 3

- ***Full domain***
- ***Completeness***
- ***Transitivity***
- ***Weak Pareto principle***
- ***Independence of irrelevant utilities***
- ***Neutrality***
- ~~***Ordinal Noncomparability***~~
- ***Cardinal Noncomparability***

Theorem (d'Aspremont & Gevers, 1977):

(CNC) is equivalent to (ONC) under (N), (IIU), and (FD).

Corollary: If $|A|>2$ and all conditions are satisfied there must be a dictator.

Possibility 1. $|N|=2$. Voting. Weighted majority rule

v_k – the number of votes voter k is allowed to cast

All v_k are arbitrary nonnegative numbers, $v_k \geq 0$, so they need not be equal.

Examples: voting of shareholders, EU states etc.

In a multi-criteria setting v_k are *weights*, reflecting the importance of criteria.

Weighted majority rule

$N(xPy) = \{ k \in N \mid u_k(x) > u_k(y) \}$ $N(yPx) = \{ k \in N \mid u_k(y) > u_k(x) \}$

$$xPy \Leftrightarrow \sum_{k \in N(xPy)} v_k > \sum_{k \in N(yPx)} v_k$$

Dictatorship: $\exists d \in N : v_d = 1$ and $v_k = 0$ for all $k \neq d$.

Simple majority rule

$$\forall k \in N,\ v_k = 1$$

Either "one person, one vote" principle or equal importance of criteria.

$xPy \Leftrightarrow |\{\ k \in N \mid u_k(x) > u_k(y)\ \}| > |\{\ k \in N \mid u_k(y) > u_k(x)\ \}|$

$xRy \Leftrightarrow (xPy \vee ((x, y) \notin P \wedge (y, x) \notin P)$, that is, R is complete by definition.

№	Journal	IF	5-IF	Immediacy index	Article influence	Hirsch	SNIP	SJR
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$$4 > 3$$

J_1 is better than J_2

Possibility 2. Nontransitive preferences

- ***Full domain***: the rule can be applied in all cases.
- ***Completeness***: all alternatives are comparable.
- ~~***Transitivity***~~
- ***Strong Pareto principle***: if $\forall k \in N$, $u_k(x) \geq u_k(y)$ then xRy;

  if also $\exists k \in N$: $u_k(x) > u_k(y)$ then xPy.
- ***Independence of irrelevant utilities***
- ***Neutrality***: the rule treats all candidates (alternatives) equally.
- ***Ordinal Noncomparability***
- ***Anonymity***: the rule treats all voters (criteria) equally.
- ***Monotonicity***: if utility profiles U and U' are such that

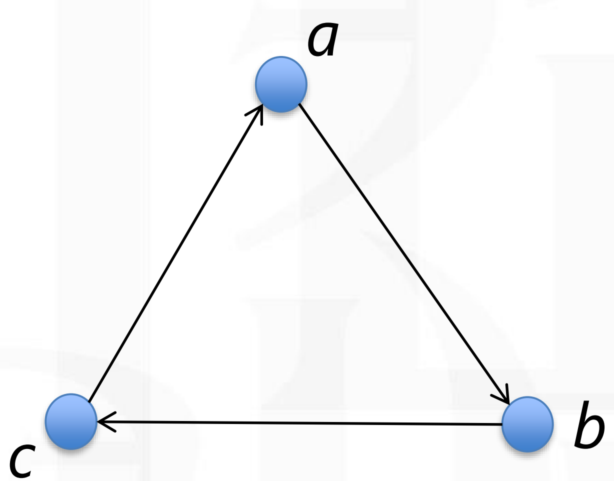
  $\forall k \in N$, $u'_k(x) \geq u_k(x) \wedge u'_k(y) = u_k(y)$ then $xPy \Rightarrow xP'y$ and $xRy \Rightarrow xR'y$.
- ***Computational simplicity***: there is a polynomial algorithm for computing R.

The Condorcet paradox (Condorcet 1785)

$$A = \{a, b, c\}\ ;\ N = \{1, 2, 3\}$$

x	$u_1(x)$	$u_2(x)$	$u_3(x)$
a	3	1	2
b	2	3	1
c	1	2	3

Utility profile U



Digraph representing P

The Condorcet paradox (real world example)

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Explorations in Economic History	**0.935**	0.898	**0.541**	0.772	7	**1.768**	**0.036**
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Scandinavian Journal of Economics	0.514	**1.070**	0.150	**1.310**	**8**	1.426	**0.043**

J_1 is better than J_2 $(4 > 3)$

J_2 is better than J_3 $(5 > 2)$

J_3 is better than J_1 $(4 > 3)$

Possibility 2a.
Nontransitive preferences and empty choices

We supposed that the choice function $C(X)$ of a *rational* actor satisfies

- ***Nonemptiness***: $\forall\ X \subseteq A,\ C(X) \neq \varnothing$;
- ***Nash Independence of irrelevant alternatives***:

$$\forall\ X \subseteq A,\ \forall\ Y \subseteq X,\ C(X) \cap Y \neq \varnothing \Rightarrow C(Y) = C(X) \cap Y.$$

But if one defines the social choice function $SC(X)$ as $MAX(R|_X)$,

where R is obtained by the simple majority rule, then

- ~~***Nonemptiness***~~ is not satisfied;
- ***Nash Independence of irrelevant alternatives*** is satisfied.

Possibility 2b. Tournament solutions

One may redefine choice function $SC(X)$ in the following way.

Let $SC(X)$ be a function of majority-rule-based nontransitive R and satisfy

- ***Nonemptiness***: $\forall\ X \subseteq A,\ SC(X) \neq \varnothing$;
- ***Neutrality***
- ***Condorcet consistency***: $MAX(R|_X) \neq \varnothing \Rightarrow SC(X)=MAX(R|_X)$

Such $SC(X)$ is called a *tournament solution*. No tournament solution satisfies

- ~~***Nash Independence of irrelevant alternatives***~~:

$$\forall\ X \subseteq A,\ \forall\ Y \subseteq X,\ C(X) \cap Y \neq \varnothing \Rightarrow C(Y) = C(X) \cap Y.$$

Theorem: there exist tournament solutions $SC(X)$ that satisfy

Weak Nash Independence of irrelevant alternatives:

$$\forall\ X \subseteq A,\ \forall\ Y \subseteq X,\ C(X) \subseteq Y \Rightarrow C(Y) = C(X).$$

Possibility 3. Domain restriction

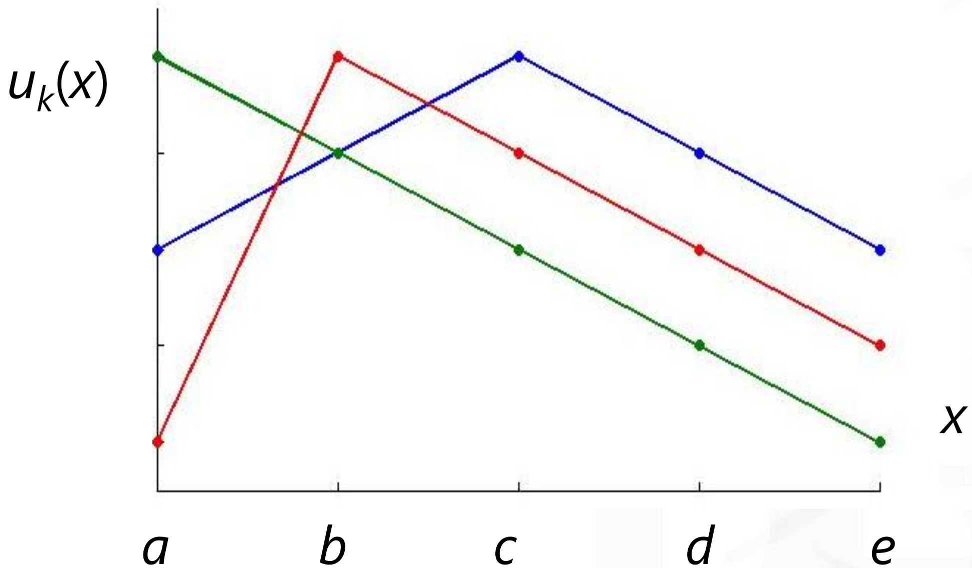
- ~~***Full domain***~~: the rule can be applied in all cases.
- ***Completeness***
- ***Transitivity***
- ***Strong Pareto principle***
- ***Independence of irrelevant utilities***
- ***Neutrality***
- ***Ordinal Noncomparability***
- ***Anonymity***
- ***Monotonicity***
- ***Computational simplicity***

Single-peaked utilities

Supposition: There is a natural linear ordering Q of the alternatives from A.

Definition: $u(x)$ is single-peaked with respect to Q if

$\exists x^* \in A$: $\forall y, z \in A$, $zQyQx^* \Rightarrow u(z)<u(y)<u(x^*)$ and $x^*QyQz \Rightarrow u(x^*)>u(y)>u(z)$.



Theorem: If all possible utility functions $u_k(x)$ are single-peaked with respect to Q then the simple majority rule will always yield a ranking R.

Possibility 4. Ordinal procedures violating independence of irrelevant alternatives

There exist many ranking procedures that preserve ordinal noncomparability at a cost of violating independence of irrelevant alternatives.

Example: ***The Copeland rule*** (1951).
Essentially, it is ranking by the number of victories won in a round-robin tournament.

1. Apply the majority rule and compute P and R.
2. For a given X count the Copeland score $s(x)$ of each $x \in X$.

version ***a*** (a tie is counted as a loss) $s_a(x) = |\{ y \in X \mid xPy \}|$

version ***b*** (a tie is counted as a victory) $s_b(x) = |\{ y \in X \mid xRy \}|$

3. Rank alternatives from X by their Copeland score.

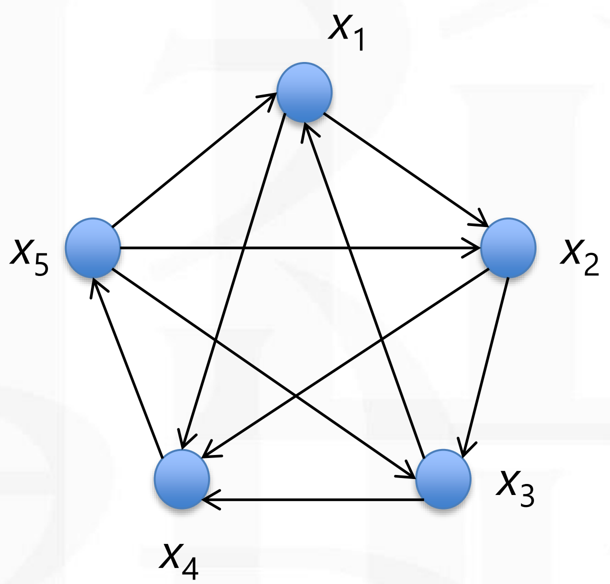
The Copeland rule. Example

M=[m_{ij}] – tournament matrix representing strict social preferences P:

$m_{xy}=1 \Leftrightarrow (x, y) \in P$, $m_{xy}=0 \Leftrightarrow (x, y) \notin P$

	x_1	x_2	x_3	x_4	x_5	*Copeland score*	*Ranking*
x_1	0	1	0	1	0	**2**	2nd best
x_2	0	0	1	1	0	**2**	2nd best
x_3	1	0	0	1	0	**2**	2nd best
x_4	0	0	0	0	1	**1**	3d best
x_5	1	1	1	0	0	**3**	the best

Tournament matrix **M**



Majority digraph

The Copeland rule. Axiomatic analysis

- ***Full domain***
- ***Completeness***
- ***Transitivity***
- ***Strong Pareto principle***
- ~~***Independence of irrelevant utilities***~~
- ***Neutrality***
- ***Anonymity***
- ***Ordinal Noncomparability***
- ***Monotonicity***
- ***Computational Simplicity***

- ***Weak Arrowian Independence of irrelevant alternatives***

Arrowian Independence of irrelevant alternatives

(AIIA) $\Leftrightarrow$ *Independence of irrelevant utilities* ∧ *Ordinal Noncomparability*

If the feasible set or the utilities change so that this does not affect the position of *x* and *y* relative to each other in any individual preference ranking, then the position of *x* relative to *y* in the social preference ranking must not change.

- ***Weak Arrowian Independence of irrelevant alternatives*** (Rubinstein 1980)

If the feasible set *stays the same*, and the utilities change so that this does not affect the position of *x* and *y* relative to each other *and to any other alternative from X* in any individual preference ranking then the position of x relative to y in the social preference ranking must not change.

(AIIA) states that the social ranking of *x* versus *y* depends only on individual ordinal binary comparisons of *x* with *y*.	(WAIIA) states that, *if the feasible choice set X does not change*, the social ranking of *x* versus *y* depends only on individual ordinal binary comparisons of *x and y with alternatives from X*.

Possibility 5. Ordinal procedures satisfying strong independence of irrelevant alternative

- ***Full domain***
- ***Completeness***
- ***Transitivity***
- ***Strong Pareto principle***
- ***Independence of irrelevant utilities***
- ***Neutrality***
- ***Anonymity***
- ~~***Ordinal Noncomparability***~~
- ***Monotonicity***
- ***Computational Simplicity***

- ***Ordinal Comparability***

The majority judgment rule

- ***Ordinal comparability***

There is a common language, that is, a common ordinal scale of evaluation.

Example: *A* ("excellent") *B* ("*good*") *C* ("satisfactory") *D* ("poor") *F* ("failed')

Students	Board of examiners					Median grade	Rating
	y_1	y_2	y_3	y_4	y_5		
x_1	*A*	*A*	*F*	*B*	*A*	***A***	the best
x_2	*B*	*B*	*C*	*F*	*A*	***B***	2nd best
x_3	*C*	*D*	*F*	*D*	*A*	***D***	4th best
x_4	*F*	*F*	*A*	*F*	*A*	***F***	5th best
x_5	*C*	*B*	*A*	*C*	*D*	***C***	3d best

x_1: *F B A A A*

^ median

General conclusion

Let

the majority

rule.

Publications

1. Subochev, A., Aleskerov, F., Pislyakov, V. 2018. Ranking journals using social choice theory methods: A novel approach in bibliometrics. *Journal of Informetrics*, 12(2), 416–429.
2. Aleskerov F., Subochev A. 2013. Modeling optimal social choice: matrix-vector representation of various solution concepts based on majority rule. *Journal of Global Optimization*, V. 56, Iss. 2. P. 737-756.

Thank you

for

your attention!

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