Topological n-valued groups

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The talk will introduce the basic concepts of the theory of topological n-valued groups X, including the construction of the Hopf n-algebra.

We will discuss the relationships of this theory with the theory of symmetric powers of spaces, and with the theory of branched coverings.

We will discus results on the action of finite groups Γ on manifolds M such that the orbit spaces M/Γ are manifolds.
We will give a derivation of the algebraic 2-valued groups addition laws on $\mathbb{C}P^1$.

D. Hilbert's Nullstellensatz (1893)

D. Hilbert, 1862–1943

Definition.

Let J be an ideal in a ring A. The radical of J is

$$\sqrt{J} = \{f \in A \mid f^n \in J \ \text{ for some } n \in \mathbb{N}\}.$$

An ideal J is said to be a radical ideal if $J = \sqrt{J}$.

Theorem.

If V is an affine algebraic variety with coordinate ring $A = \mathbb{C}[x_1, \dots, x_n]/J$, where J is a (radical) ideal defining V, then the evaluation map

$$\mathcal{E} : V \to \mathrm{Hom}(A, \mathbb{C}), \quad \mathcal{E}(x)\varphi = \varphi(x),$$

defines a homeomorphism between the variety V and the set of all ring homomorphisms $A \to \mathbb{C}$.

A.N.Kolmogorov's and I.M.Gel'fand's theorem (1939)

A.N.Kolmogorov, 1903–1987

I.M.Gel'fand, 1913–2009

Theorem.

Let X be a compact Hausdorff space. Then, for an appropriate topology on the space of continuous complex-valued functions $C(X)$ on X, the evaluation map

$$\mathcal{E} : X \to \mathrm{Hom}(\mathrm{C}(\mathrm{X}), \mathbb{C}), \quad \mathcal{E}(\mathrm{x})\varphi = \varphi(\mathrm{x}),$$

is a homeomorphism onto the set of all ring homomorphisms $C(X) \to \mathbb{C}$.

Symmetric products

The symmetric product of a space X is the quotient space

$$(X)^n = X^n/\Sigma_n = \{(x_1, \ldots, x_n) : (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \sim (x_1, \ldots, x_n),\ \sigma \in \Sigma_n\},$$

where Σ_n is the group of all permutations of a set with n elements.

The continuous functions on $(X)^n$ correspond exactly to the continuous functions $f : X^n \to \mathbb{C}$ invariant under all permutations of the coordinates, that is, the symmetric functions.

Let us consider an analogue of the evaluation map

$$\mathcal{E} : (X)^n \longrightarrow \mathrm{Hom}(\mathrm{C}(\mathrm{X}), \mathbb{C}), \quad \mathcal{E}(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n})(\varphi) = \varphi(\mathrm{x}_1) + \cdots + \varphi(\mathrm{x}_\mathrm{n}).$$

We obtain a functorial embedding of the space $(X)^n$ into the linear space $\mathrm{Hom}(\mathrm{C}(\mathrm{X}), \mathbb{C})$.

Let V_g be a non-singular algebraic curve of genus $g \geqslant 0$.
For any $k \geqslant 1$, the symmetric degree $(V_g)^k$ is a non-singular projective algebraic manifold.

Example. The spaces $(\mathbb{C})^n = \mathbb{C}^n/\Sigma_n$ and $\mathbb{C}^n$ can be identified using the map

$$\mathcal{S} : \mathbb{C}^n \to \mathbb{C}^n : \ (z_1, z_2, \dots, z_n) \to e_r(z_1, z_2, \dots, z_n), \ 1 \leqslant r \leqslant n,$$

where e_r is the r-th elementary symmetric polynomial.

The projectivization of the map $\mathcal{S}$ induces the homeomorphism $(\mathbb{CP}^1)^n \to \mathbb{CP}^n$.

In the paper

V.M.Buchstaber, E.G.Rees,
Multivalued groups and Hopf n-algebras.
Russian Math. Surveys, 51:4, 1996, 727–729.

it was given the definition of the algebraic n-homomorphism $A \to B$, where A and B be two associative $\mathbb{C}$-algebras with the units.
In terms of such a homomorphism, it was introduced the concept of the structure of a Hopf n-algebra.

In the paper

V.M.Buchstaber, E.G.Rees,
Frobenius k-characters and n-ring homomorphisms.
Russian Math. Surveys, 52:2 (1997), 398–399.

it was given the definition of the Frobenius n-homomorphism $A \to B$ and it was proved that it is equivalent to our definition of an algebraic n-homomorphism.

Using this concept, we obtained a description of the space $(X)^n \subset \mathrm{Hom}(\mathrm{C(X)}, \mathbb{C})$ for any Hausdorff space X and $n \in \mathbb{N}$.

Main recursion

Let A and B be two associative $\mathbb{C}$-algebras with the units, where B is a commutative algebra.

Let $f : A \to B$ be a linear trace-like map, i.e. a map such that $f(a_1 a_2) = f(a_2 a_1)$ for every $a_1, a_2 \in A$.

We introduce linear maps $\Phi_n(f) : A^{\otimes n} \to B,\ n \in \mathbb{N}$, by setting

$$\Phi_1(f) = f, \quad \Phi_2(f)(a_1 \otimes a_2) = f(a_1)f(a_2) - f(a_1 a_2),$$

and further by recurrence,

$$\begin{aligned}\Phi_{n+1}(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) &= f(a_1)\Phi_n(f)(a_2 \otimes \cdots \otimes a_{n+1}) - \\ &- \Phi_n(f)(a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1}) - \cdots - \Phi_n(f)(a_2 \otimes a_3 \otimes \cdots \otimes a_1 a_{n+1}).\end{aligned}$$

Frobenius n-homomorphism

Definition.

By a Frobenius n-homomorphism we mean a linear trace-like homomorphism $f : A \to B$ satisfying the conditions $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

Examples.

1-homomorphism: $f(1) = 1$ and $f(a_1 a_2) = f(a_1) f(a_2)$,

that is, a ring homomorphism.

2-homomorphism: $f(1) = 2$ and

$$2f(a_1 a_2 a_3) = f(a_1) f(a_2 a_3) + f(a_1 a_2) f(a_3) + f(a_2) f(a_1 a_3) - f(a_1) f(a_2) f(a_3).$$

Theorem. (Buchstaber, Rees, 2002)

Let X be a compact Hausdorff space. Then the evaluation map

$$\mathcal{E} : (X)^n \to \mathrm{Hom}(\mathrm{C}(\mathrm{X}), \mathbb{C}), \quad \mathcal{E}(\mathrm{x_1}, \dots, \mathrm{x_n})(\varphi) = \varphi(\mathrm{x_1}) + \dots + \varphi(\mathrm{x_n}),$$

is a homeomorphism onto the set of all Frobenius n-homomorphisms $f\colon C(X) \to \mathbb{C}$, i.e. it is given by the equations $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

Theorem. (Buchstaber, Rees, 2002)

Let A be a finitely generated commutative algebra and let V be the affine algebraic variety $\mathrm{m\text{-}Spec}(A)$. Then the map

$$\mathcal{E} : (V)^n \to \mathrm{Hom}(\mathrm{A}, \mathbb{C}), \quad \mathcal{E}(\mathrm{f_1}, \dots, \mathrm{f_n})(\mathrm{a}) = \mathrm{f_1}(\mathrm{a}) + \dots + \mathrm{f_n}(\mathrm{a}),$$

where $f_k : A \to \mathbb{C}$, $k = 1, \dots, n$, are ring homomorphisms (whose kernels are the maximal ideals which are points of V), defines a homeomorphism

$$\mathcal{E} : (V)^n \longrightarrow \Phi_n(A)$$

where $\Phi_n(A)$ is the set of all Frobenius n-homomorphisms $A \to \mathbb{C}$.

Solution of the recursion

Let $\mathcal{D}_f(a_1, \dots, a_{n+1})$ be the determinant of the matrix

$$\begin{pmatrix} f(a_1) & 1 & \dots & 0 \\ f(a_1 a_2) & f(a_2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ f(a_1 \cdots a_n) & f(a_2 \cdots a_n) & \dots & n \\ f(a_1 \cdots a_{n+1}) & f(a_2 \dots a_{n+1}) & \dots & f(a_{n+1}) \end{pmatrix}$$

Theorem. (V.M.Buchstaber, E.Rees, 1997)

$$\Phi_{n+1}(f)(a_1, \dots, a_{n+1}) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \mathcal{D}_f\left(a_{\sigma(1)}, \dots, a_{\sigma(n+1)}\right).$$

Corollary.

The map $\Phi_{n+1}(f)$ is a symmetric multi-linear form.

Multiplicative property

We denote the sub-algebra of symmetric tensors in $A^{\otimes n}$ by $\mathcal{S}^n A$.

A typical element of $\mathcal{S}^n A$ is

$$\mathbf{a} = \sum_{\sigma \in \Sigma_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

and

$$\mathbf{ab} = \sum_{\sigma_1, \sigma_2 \in \Sigma_n} a_{\sigma_1(1)} b_{\sigma_2(1)} \otimes \cdots \otimes a_{\sigma_1(n)} b_{\sigma_2(n)}.$$

Theorem. (V.M.Buchstaber, E.Rees)

The linear map $f: A \to B$ is a Frobenius n-homomorphism if and only if

$$\frac{1}{n!}\Phi_n(f): \mathcal{S}^n A \to B$$

is a ring homomorphism.

Branched coverings

We consider branched coverings in the sense studied by Smith and Dold.

Definition.

An n-branched covering $h : X \to Y$ is a continuous map between two Hausdorff spaces with a continuous map $t : Y \to (X)^n$ such that

(i) $x \in th(x)$ for every $x \in X$;

(ii) $Y \longrightarrow (X)^n \longrightarrow (Y)^n, \quad (h)^n(ty) = ny$ for every $y \in Y$.

Example.

The map $p : \mathbb{C} \to \mathbb{C}$ defined by a polynomial of degree n is the classic example of an n-branched covering.

The map t is defined by $t(w) = [z_1, z_2, \ldots, z_n]$ where the z_r are the roots (counted with multiplicities) of the equation $p(z) = w$.

It is straightforward to verify the above axioms in this case.

Finite group actions

Let G be a finite group acting continuously and effectively on a Hausdorff space X and let $h\colon X \to Y = X/G$ be the map to the space of orbits.

Then h is an n-branched covering where n is the cardinality of G and the map $t\colon Y \to (X)^n$ is given by $t(y) =$ the points in the orbit corresponding to y (counted with multiplicities).

Let $H \subset G$ be a subgroup of finite index n and let X be an effective G-space. Then the quotient map $X/H \to X/G$ is an n-branched covering.

Every 2-branched covering arises from an action of the group with two elements.

Theorem. (A.Dold, 1986)

Every n-branched covering $h\colon X \to Y$ can be described in the form of a projection

$$p\colon E\times_{\Sigma_n}[n] \to E/\Sigma_n$$

for the Σ_n-space E, where $[n] = \{1, \dots, n\}$ and

$$E = \{\varphi(y;\cdot)\colon [n] \to X \;:\; h\varphi(y;k) = y,\ k \in [n]\}.$$

Branched coverings of manifolds

Definition.

Let X and Y be two connected topological m-dim manifolds without boundary, $m \geqslant 2$.
A continuous map $f : X \to Y$ is called a finite-sheeted branched covering if it is open, closed, and each point has only finitely many preimages.

Any finite-sheeted branched covering of manifolds $f : X \to Y$ is a branched covering in sense of Smith–Dold.

Moreover, for such a map f its n-valued inverse $t : Y \to (X)^n$ can be constructed in a unique natural way.

The main structure theorem about such mappings was proved by A.V.Chernavskii (1964).

Frobenius n-homomorphisms and transfer maps

Definition.

Let A, B be commutative, associative $\mathbb{C}$-algebras and let $f : A \to B$ be a ring homomorphism. Then a linear map $\tau : B \to A$ is an n-transfer for f if:

(i) τ is a Frobenius n-homomorphism;

(ii) $\tau(f(a)b) = a\tau(b)$, that is τ is a map of A-modules;

(iii) $f\tau : B \to B$ is the sum of the identity and a Frobenius $(n-1)$-homomorphism $g : B \to B$.

When X is a compact Hausdorff space, $C(X)$ will denote the algebra of continuous functions $X \to \mathbb{C}$ with the supremum norm.

Definition.

The direct image $f_! : C(X) \to C(Y)$ associated with a continuous map $t : Y \to (X)^n$ is defined by $(t_! \varphi)(y) = \sum \varphi(x_r)$ where $t(y) = [x_1, x_2, \ldots, x_n]$.

Theorem.

Let X, Y be compact Hausdorff spaces. Then the set of all continuous Frobenius n-homomorphisms $C(X) \to C(Y)$ can be identified with the space of continuous maps $Y \to (X)^n$

Theorem.

Let $h : X \to Y$ with $t : Y \to (X)^n$ be an n-branched covering. Then direct image $t_! : C(X) \to C(Y)$ is an n-transfer for the ring homomorphism $h^* : C(Y) \to C(X)$.

Theorem.

Let $h : X \to Y$ be a continuous map, and let τ for $h^* : C(Y) \to C(X)$ be a continuous n-transfer. Then h is an n-branched covering.

Theorem.

Let A, B be finitely generated commutative algebras and $f : A \to B$ – a ring homomorphism.
Let $V(A), V(B)$ be the corresponding algebraic varieties and $h : V(B) \to V(A)$ – the map corresponding to f.
Then h is an n-branched covering if and only if there is an n-transfer $B \to A$, for h.

n-valued group structure

An n-valued multiplication on a topological space X is a continuous map

$$\mu : X \times X \to (X)^n, \quad \mu(x,y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x*y)_k.$$

Associativity. The n^2-sets:

$$[x*(y*z)_1, x*(y*z)_2, \dots, x*(y*z)_n], \quad [(x*y)_1*z, (x*y)_2*z, \dots, (x*y)_n*z]$$

are equal for all $x, y, z \in X$.

Unit. An element $e \in X$ such that $\quad e*x = x*e = [x, x, \dots, x] \;$ for all $x \in X$.

Inverse. A map $\mathrm{inv}\colon X \to X$ such that $\quad e \in \mathrm{inv}(x)*x \quad$ and $\quad e \in x*\mathrm{inv}(x)$ for all $x \in X$.

Definition.

The map μ defines an n-valued topological group structure $\mathcal{X} = (X, \mu, e, \mathrm{inv})$ on X if it is associative, has a unit, and an inverse.

An n-valued group structure on X is commutative if $x * y = y * x$.

Definition.

A map $f \colon X \to Y$ is a homomorphism of n-valued groups if

$$f(e_X) = e_Y, \quad f(\mathrm{inv}_X(x)) = \mathrm{inv}_Y(f(x)) \quad \text{for all } x \in X$$

and

$$\mu_Y(f(x_1), f(x_2)) = (f)^n \mu_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Homomorphisms of n-valued groups

Lemma.

Let $f\colon X \to Y$ be a homomorphism of n-valued groups. Then:

- $\mathrm{Ker}(f) = \{x \in X \mid f(x) = e_Y\}$ is an n-valued group.
- $f(x_1) = f(x_2)$ if and only if $(f)^n(zx_1) = (f)^n(zx_2)$ for all $z \in \mathrm{Ker}(f)$.
- Let the mapping $\mathrm{inv}\colon X \to X$ be uniquely defined. Then $\mathrm{Ker}(f) = \{e\}$ if and only if equality $x_1 = x_2$ follows from equality $f(x_1) = f(x_2)$.
- $\mathrm{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$ is an n-valued group.

Coset groups

The following construction produces many n-valued groups.
Let G be a (1-valued) group with the multiplication μ_0, the unit e_G, and $\mathrm{inv}_G(u) = u^{-1}$.
Let A be a group with $\#A = n$ and $\varphi : A \to Aut\ G$ be a homomorphism to the group of automorphisms of G.
Denote by X the quotient space $G/\varphi(A)$ of G by the action of the group $\mathrm{Im}\varphi$, and denote by $\pi\colon G \to X$ the quotient map. Define the n-valued multiplication

$$\mu\colon X \times X \to (X)^n$$

by the formula $\mu(x, y) = [\pi(\mu_0(u, v^{a_i})), \quad 1 \leqslant i \leqslant n, \quad a_i \in A]$,
where $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$ and v^a is the image of the action of $\varphi(a) \in Aut\ G$, $a \in A$ on G.

Theorem.

The multiplication μ defines an n-valued group structure on the orbit space $X = G/\varphi(A)$, called the coset group of (G, A, φ), with the unit $e_X = \pi(e_G)$ and the inverse $\mathrm{inv}_X(x) = \pi(\mathrm{inv}_G(u))$, where $u \in \pi^{-1}(x)$.

First examples of topological n-valued groups

Set $\varepsilon = \exp(2\pi i/n)$. The automorphism $\varepsilon\colon \mathbb{C} \to \mathbb{C} : \varepsilon(z) = \varepsilon z$ define the multiplication $\mu\colon \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^n$ in n-valued coset group by the formula

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y}\,)^n, \quad 1 \leqslant r \leqslant n].$$

The unit: $e = 0$. The inverse: $\mathrm{inv}(x) = (-1)^n x$.

The multiplication is described by the polynomial equations with integer coefficients

$$p_n(x,y,z) = \prod_{k=1}^{n} \left(z - (x*y)_k\right) = 0.$$

For instance,

$$\begin{aligned} p_1 &= z - x - y, \\ p_2 &= (z+x+y)^2 - 4(xy + yz + zx), \\ p_3 &= p_1^3 - 27xyz, \\ p_4 &= p_2^2 - 2^7(z+x+y)xyz. \end{aligned}$$

Examples of the coset groups

Consider the n-fold direct product G^n of a group G by itself.
The group Σ_n acts on G^n by permuting the factors.

Therefore, for any group G, the symmetric product $(G)^n$ is endowed with the structure of an $n!$-valued coset group.

If G is commutative, with the operation $\mu(g', g'') = g' + g''$, then we have an $n!$-valued group homomorphism

$$(\mu)^n : (G)^n \longrightarrow G, \quad [g_1, \dots, g_n] \longrightarrow g_1 + \dots + g_n,$$

where G is treated as an $n!$-valued group with the diagonal operation $\mu(g', g'') = [g' + g'', \dots, g' + g'']$.

In this way we obtain the $n!$-valued group $\mathrm{Ker}(\mu)^n$.

Take a smooth elliptic curve. It equips the torus T^2
with a commutative group structure.
The construction above produces a structure
of an $n!$-valued group on $(T^2)^n$ for each n.

Thus this construction produces a structure
of an $n!$-valued group on the complex projective space

$$\mathbb{C}P^{n-1} = \mathrm{Ker}((\mu)^n : (T^2)^n \to T^2).$$

For $n = 2$, this yields a structure of a 2-valued group on $\mathbb{C}P^1$.

Topological n-valued groups

Classical result

Let $A \subset \mathsf{Aut}_+(\mathbb{T}^2) = SL(2,\mathbb{Z})$ be a finite subgroup of the automorphisms group of the torus $\mathbb{T}^2$, that preserve its orientation.
Then $A = \mathbb{Z}/n$, where $n \in \{2,3,4,6\}$.

Theorem.

An automorphism α of a smooth elliptic curve T^2 of order $n \in \{2,3,4,6\}$ defines the structure of an n-valued coset group on the complex projective line $\mathbb{C}P^1$.

Orbit spaces of finite groups action

Theorem (M.A. Mikhailova, 1985)

Let M^3 be a smooth closed oriented manifold and G – a finite group acting on M^3 by orientation-preserving diffeomorphisms.
Then the orbit space M^3/G is a topological manifold.

Theorem (D.V. Gugnin, 2023)

Let an orientation-preserving action of a finite group G on the sphere S^3 have a fixed point.
Then the orbit space S^3/G is homeomorphic to the sphere S^3.

Proof of Gugnin's theorem uses the result of G.Ya. Perelman that a three-dimensional closed simply connected manifold is homeomorphic to the sphere S^3.

Orbit spaces of finite groups action

An orientation-preserving orthogonal transformation that is identical on some subspace of codimension 2, is called a rotation of the space $\mathbb{R}^n$.

Theorem (C. Lange, M.A. Mikhailova, 2016)

Let $G \subset SO(n)$ be a finite group generated by rotations.
Then the orbit space $\mathbb{R}^n/G$ is homeomorphic to the space $\mathbb{R}^n$.

Corollary.

If $G \subset SO(n)$ is a finite group generated by rotations,
then the structure of a $|G|$-valued commutative cosset group is defined on the space $\mathbb{R}^n \simeq \mathbb{R}^n/G$.

Double coset groups

Let $H \subset G$ be a subgroup, and let $\#H = n$. Denote by X the space of double cosets $H\backslash G/H$. Define the n-valued multiplication

$$\mu\colon X \times X \to (X)^n$$

by the formula $\mu(x, y) = \{Hg_1H\} * \{Hg_2H\} = [\{Hg_1hg_2H\} : \ h \in H]$.

Theorem.

The multiplication μ defines an n-valued group structure on the orbit space $X = H\backslash G/H$ (called a double coset group) of a pair (G, H) with the unit $e_X = \{H\}$ and the inverse $\mathrm{inv}_X(x) = \{Hg^{-1}H\}$, where $x = \{HgH\}$.

Each coset group $X = G/\varphi(A)$, where $\varphi : A \to Aut\ G$ is a homomorphism to the group of automorphisms of G, admits a double coset realization $X = A\backslash G'/A$, where G' is the semidirect product of the groups G and A with respect to the action of A on G by means of φ.

n-Hopf algebras and Frobenius n-homomorphisms

Associate to a space X the ring of functions $C(X)$. For any positive integer k, we have the canonical n-homomorphisms

$$s_k : C(X) \longrightarrow C((X)^k) \text{ such that } s_k(f)[x_1, \dots, x_k] = \sum_{i=1}^{k} f(x_i).$$

Definition.

Let X be an n-valued group with the multiplication $\mu : X \times X \longrightarrow (X)^n$. The diagonal map

$$\Delta : C(X) \longrightarrow C(X \times X) \approx C(X) \widehat{\otimes} C(X)$$

is the linear map $\Delta = \frac{1}{n} F$, where $F(f)(x, y) = s_n(f)(\mu(x, y)) = \sum_{i=1}^{n} f(z_i)$ and $\mu(x, y) = x * y = [z_1, \dots, z_n]$.

The $\mathrm{inv}: X \to X$ map axiom implies that there is a map $\mathrm{inv}^{\perp}: X \to (X)^{n-1}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[drr, "\mathrm{inv}^{\perp}"'] & X \times X \arrow[r, "1\times \mathrm{inv}"] & X \times X \arrow[r, "\mu"] & (X)^n \\
& & (X)^{n-1} \arrow[ur, "i_n"'] &
\end{tikzcd}$$

is commutative; here $i_n[x_1, \dots, x_{n-1}] = [x_1, \dots, x_{n-1}, e]$.

This diagram states that the homomorphism $s_n(\cdot)(\mu(x, \mathrm{inv}\, x)) : \mathbb{C}[X] \to \mathbb{C}[X]$ is split into composition of homomorphisms $(\mathrm{inv}^{\perp})^* \, i_n^* \, s_n(\cdot)$.

If $n = 1$, then this algebraic condition determines the antipode

$$(\mathrm{inv})^* : \mathbb{C}[X] \to \mathbb{C}[X]$$

in the Hopf algebra $\mathbb{C}[X]$.

Definition.

Let A be a commutative algebra over a ring k containing $\frac{1}{n}$.
We say that A is endowed with an n-bialgebra structure (A, Δ, ε) if

1. $\varepsilon : A \to k$ is a ring homomorphism;
2. $\Delta : A \to A \otimes A$ is a linear homomorphism making A a coalgebra (A, Δ, ε), i.e., the coassociativity axioms for Δ and counit axioms for ε are satisfied;
3. the linear homomorphism $F = n\Delta : A \to A \otimes A$ is a Frobenius n-homomorphism.

Definition.

An n-bialgebra (A, Δ, ε) is an n-Hopf algebra $(A, \Delta, \varepsilon, \chi, \chi^{\perp})$ if

1. $\chi : A \longrightarrow A$ is a ring homomorphism;
2. $\chi^{\perp} : A \to A$ is a Frobenius $(n-1)$-homomorphism such that the Frobenius n-homomorphism $A \xrightarrow{F} A \otimes A \xrightarrow{1\otimes\chi} A \otimes A \xrightarrow{m} A$ splits into a sum $(\eta\varepsilon) + \chi^{\perp}$, where m is the multipluication in A and $\eta : k \to A$ is the structure map defining $1 \in A$.

Theorem.

Let X be an n-valued group. Then the ring $C(X)$ carries a structure of an n-Hopf algebra $(C(X), \Delta, \varepsilon, \chi, \chi^{\perp})$, where Δ is the diagonal homomorphism introduced above, ε, χ are ring homomorphisms induced by the maps $e \to X$ and $\mathrm{inv} : X \to X$, respectively, and $\chi^{\perp}$ is the $(n-1)$-homomorphism $(\mathrm{inv}^{\perp})^{*} s_{n-1}$.

Theorem.

If X is a compact Hausdorff space and the ring of functions $C(X)$ is endowed with an n-Hopf algebra structure $(C(X), \Delta, \varepsilon, \chi, \chi^{\perp})$, then X is an n-valued group.

Cohomology rings with the structures on n-Hopf algebras

Theorem.

Let X be a topological n-valued group. Then the algebra $H^{2*}(X; Q)$ carries an induced structure of an n-Hopf algebra.

As a consequence, one can prove that the spaces $\mathbb{C}P^m$ for $m > 1$ do not admit the structure of a 2-valued group. Any structure of an elliptic curve on torus T^2 gives the structure of coset 2-valued group $T^2/(\pm 1)$ on $\mathbb{C}P^1$.

Another important corollary of this result:

Theorem. (T.E.Panov, 1996)

The only (up to homotopy equivalence) simply connected 4-dimensional manifolds admitting a structure of 2-Hopf algebra in the cohomology algebra are $k\mathbb{C}P^2\#(6-k)(-\mathbb{C}P^2)$ and $3(S^2 \times S^2)$.

Here $\#$ denotes the connected sum of manifolds,
the "$-$"sign before $\mathbb{C}P^2$ means the inversion of orientation,
and kM denotes the connected sum of k copies of a manifold M.

Elliptic functions

The affine part of the elliptic curve V in the Weierstrass model has the form

$$\{(x, y) \in \mathbb{C}^2 \ : \ y^2 = 4x^3 - g_2 x - g_3\}$$

where g_2, $g_3 \in \mathbb{C}$. The curve V is nonsingular iff $g_2^3 \neq 27 g_3^2$.

A nonsingular elliptic curve is homeomorphic to the torus $T^2 = \mathbb{C}/\Gamma$, where Γ is a lattice with generators $2\omega_1,\ 2\omega_2 \in \mathbb{C}$ and $\omega_1 \neq \lambda\omega_2,\ \lambda \in \mathbb{R}$.

A function $f(z),\ z \in \mathbb{C}$, is called doubly periodic if

$$f(z + 2\omega_1) = f(z), \quad f(z + 2\omega_2) = f(z).$$

A doubly periodic analytic function that has no singular points other than poles in $\mathbb{C}$ is called an elliptic function.

Generators of elliptic functions field

The Weierstrass elliptic function $\wp(z)$ is uniquely defined as a solution to the differential equation

$$(\wp')^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$$

with the initial condition $\wp(z) = \frac{1}{z^2} + o(z)$.

The function $\wp(z)$ is an even function and $\wp'(z)$ is an odd function.

Theorem.

Any elliptic function $f(z)$ with periods $2\omega_1$ and $2\omega_2$ can be expressed in terms of functions $\wp(z)$ and $\wp'(z)$ with the same periods, and these expressions are rational in $\wp(z)$ and linear in $\wp'(z)$.

Addition Theorem.

$$\wp(u+v) + \wp(u-v) = (\wp(u) - \wp(v))^{-2}\Big[(2\wp(u)\wp(v) - \frac{1}{2}g_2)(\wp(u) + \wp(v)) - g_3\Big];$$

$$\wp(u+v)\wp(u-v) = (\wp(u) - \wp(v))^{-2}\Big[(\wp(u)\wp(v) + \frac{1}{4}g_2)^2 + g_3(\wp(u) + \wp(v))\Big].$$

Structure of an algebraic 2-valued group on $\mathbb{C}P^1$

Let z be a coordinate in $\mathbb{C}$ and u be a coordinate in $T^2 = \mathbb{C}/\Gamma$. Consider the automorphism $\tau\colon T^2 \to T^2$, $\tau(u) = -u$. We obtain the structure of a coset 2-valued group on $T^2/\tau \cong S^2$.

The entire function $\sigma(z) = z + o(z)$ (the Weierstrass sigma function) is uniquely defined by

$$\wp(z) = -\frac{d^2 \ln \sigma(z)}{dz^2} = \sigma^{-2}(z)\Big((\sigma')^2(z) - \sigma''(z)\sigma(z)\Big),$$

where $\sigma(z + 2\omega_k) = -\exp(2\eta_k(z + \omega_k))\sigma(z)$, $\eta_k \in \mathbb{C}$, $k = 1, 2$.

For a non-singular elliptic curve V the holomorphic map

$$\wp\colon V \to \mathbb{C}P^1 \ :\ \wp(u) = [(\sigma')^2(z) - \sigma''(z)\sigma(z) \ :\ \sigma^2(z)]$$

can be decomposed into a composition with a homeomorphism $T^2/\tau \to \mathbb{C}P^1$.

Algebraic addition law

Set $x = \wp(u)$, $y = \wp(v)$. We have $x * y = [\wp(u+v), \wp(u-v)]$.

Consider the equation $\Theta_0(x,y)z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0$ where

$$\frac{\Theta_1(x,y)}{\Theta_0(x,y)} = \wp(u+v) + \wp(u-v), \qquad \frac{\Theta_2(x,y)}{\Theta_0(x,y)} = \wp(u+v)\wp(u-v).$$

Using classical Addition Theorem for $\wp(u)$, we obtain

$$\Theta_0(x,y) = (x-y)^2; \qquad \Theta_1(x,y) = \big(2xy - \frac{1}{2}g_2\big)(x+y) - g_3;$$
$$\Theta_2(x,y) = \big(xy + \frac{1}{4}g_2\big)^2 + g_3(x+y).$$

The condition $\Theta_0 = \Theta_1 = \Theta_2 = 0$ is equivalent to the condition $g_2^3 = 27g_3^2$.

Theorem.

The mapping $\wp\colon V \to \mathbb{C}P^1$ defines the algebraic addition law

$$\mu\colon \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2 \ : \ x * y = (\Theta_0 : \Theta_1 : \Theta_2)$$

of 2-valued coset group on $\mathbb{C}P^1$.

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THANK YOU FOR ATTENTION!